

Stochastic invariance of closed sets for jump-diffusions with non-Lipschitz coefficients

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December 23, 2016

Abstract

We provide necessary and sufficient first order geometric conditions for the stochastic invariance of a closed subset of \mathbb{R}^d with respect to a jump-diffusion under weak regularity assumptions on the coefficients. Our main result extends the recent characterization proved in Abi Jaber, Bouchard and Illand (2016) to jump-diffusions. We also derive an equivalent formulation in the semimartingale framework.

Keywords: Stochastic differential equation, jumps, semimartingale, stochastic invariance.

MSC Classification: 93E03, 60H10, 60J75.

1 Introduction

We consider a weak solution to the following stochastic differential equation with jumps

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int \rho(X_{t-}, z) (\mu(dt, dz) - F(dz)dt), \quad X_0 = x, \quad (1.1)$$

that is: a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F})_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions and supporting a d -dimensional Brownian motion W , a Poisson random measure μ on $\mathbb{R}_+ \times \mathbb{R}^d$ with compensator $dt \otimes F(dz)$, and a \mathbb{F} -adapted process X with càdlàg sample paths such that (1.1) holds \mathbb{P} -almost surely.

Throughout this paper, we assume that $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{M}^d$ and $\rho : \mathbb{R}^d \times \mathbb{R}^d \mapsto \mathbb{R}^d$ are measurable, where \mathbb{M}^d denotes the space of $d \times d$ matrices. In addition, we assume that

$$b, \sigma \text{ and } \int \rho(., z)^\top H(\rho(., z)) \rho(., z) F(dz) \text{ are continuous for any } H \in \mathcal{C}_b(\mathbb{R}^d, \mathbb{M}^d), \quad (H_C)$$

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where $\mathcal{C}_b(\mathbb{R}^d, \mathbb{M}^d)$ denotes the space of \mathbb{M}^d -valued continuous bounded functions. We also assume that there exist $q, L > 0$ such that, for all $x \in \mathcal{D}$,

$$\int_{\{\|\rho(x,z)\|>1\}} \|\rho(x,z)\|^q \ln \|\rho(x,z)\| F(dz) \leq L(1 + \|x\|^q), \quad (H_0)$$

$$\|b(x)\|^2 + \|\sigma\sigma^\top(x)\| + \int \|\rho(x,z)\|^2 F(dz) \leq L(1 + \|x\|^2). \quad (H_1)$$

Let \mathcal{D} denote a closed subset of \mathbb{R}^d . Our aim is to characterize the stochastic invariance (a.k.a viability) of \mathcal{D} under weak regularity assumptions, i.e. find necessary and sufficient conditions on the coefficients such that, for all $x \in \mathcal{D}$, there exists a \mathcal{D} -valued solution to (1.1) starting at x .

Invariance and viability problems have been intensively studied in the literature, first in a deterministic setup [2] and later in a random environment. For the diffusion case, see [1, 9, 4] and the references therein. In the presence of jumps, we refer to [20, 22, 13]. Note that a first order characterization for a smooth volatility matrix σ is given in [13], where the Stratonovich drift appears (see [9] for the diffusion case). For a second order characterization, we refer to [22, Propositions 2.13 and 2.15].

Combining the techniques used in [1, 22], we derive for the first time in Theorem 2.2 below, a first order geometric characterization of the stochastic invariance with respect to (1.1) when the volatility matrix σ can fail to be differentiable. We also provide an equivalent formulation of the stochastic invariance with respect to a (special) semimartingale in Theorem 3.1. This extends [1] to the jump-diffusion case. From a practical perspective, this is the first known first order characterization that could be directly applied to construct affine [10, 16] and polynomial processes [8] on any arbitrary closed sets, since for these processes the volatility matrix can fail to be differentiable (on the boundary of the domain).

In fact, in the sequel, we only make the following assumption on the covariance matrix

$$C := \sigma\sigma^\top \text{ on } \mathcal{D} \text{ can be extended to a } C_{loc}^{1,1}(\mathbb{R}^d, \mathbb{S}^d) \text{ function,} \quad (H_2)$$

in which $\mathcal{C}_{loc}^{1,1}$ means \mathcal{C}^1 with a locally Lipschitz derivative and \mathbb{S}^d denotes the set of $d \times d$ symmetric matrices. Note that we do not impose the extension of C to be positive semi-definite outside \mathcal{D} , so that σ might only match with its square-root on \mathcal{D} . Also, it should be clear that the extension needs only to be local around \mathcal{D} .

From now on we use the same notation C for C defined as $\sigma\sigma^\top$ on \mathcal{D} and for its extension defined in Assumption (H₂). All identities involving random variables have to be considered in the a.s. sense, the probability space and the probability measure being given by the context. Elements of \mathbb{R}^d are viewed as column vectors. We use the standard notation I_d to denote the $d \times d$ identity matrix and denote by \mathbb{M}^d the collection of $d \times d$ matrices. We say that $A \in \mathbb{S}^d$ (resp. \mathbb{S}_+^d) if it is a symmetric (resp. and positive semi-definite) element of \mathbb{M}^d . Elements of \mathbb{R}^d are viewed as column vectors. Given $x = (x^1, \dots, x^d) \in \mathbb{R}^d$, $\text{diag}[x]$ denotes the diagonal matrix whose i -th diagonal component is x^i . If A is a symmetric positive semi-definite matrix, then $A^{\frac{1}{2}}$ stands for its symmetric square-root.

The rest of the paper is organized as follows. Our main result is stated and proved in Section 2. An equivalent formulation in the semimartingale framework is derived in Section 3. In the Appendix, we adapt to our setting some technical results, mainly from [1].

2 Main result

We start by making precise the definition of stochastic invariance.

Definition 2.1 (Stochastic invariance). A closed subset $\mathcal{D} \subset \mathbb{R}^d$ is said to be stochastically invariant with respect to the jump-diffusion (1.1) if, for all $x \in \mathcal{D}$, there exists a weak solution X to (1.1) starting at $X_0 = x$ such that $X_t \in \mathcal{D}$ for all $t \geq 0$, almost surely.

The following theorem provides a first order geometric characterization of the stochastic invariance using the (first order) normal cone $\mathcal{N}_{\mathcal{D}}(x)$ at x consisting of all outward pointing vectors,

$$\mathcal{N}_{\mathcal{D}}(x) := \left\{ u \in \mathbb{R}^d : \langle u, y - x \rangle \leq o(\|y - x\|), \forall y \in \mathcal{D} \right\}.$$

Theorem 2.2. Let $\mathcal{D} \subset \mathbb{R}^d$ be closed. Under the continuity assumptions (H_C) and (H_0) – (H_2) , the set \mathcal{D} is stochastically invariant with respect to the jump-diffusion (1.1) if and only if

$$\begin{cases} x + \rho(x, z) \in \mathcal{D}, \text{ for } F\text{-almost all } z, & (2.1a) \\ \int |\langle u, \rho(x, z) \rangle| F(dz) < \infty, & (2.1b) \\ C(x)u = 0, & (2.1c) \\ \langle u, b(x) - \int \rho(x, z) F(dz) - \frac{1}{2} \sum_{j=1}^d DC^j(x)(CC^+)^j(x) \rangle \leq 0, & (2.1d) \end{cases}$$

for all $x \in \mathcal{D}$ and $u \in \mathcal{N}_{\mathcal{D}}(x)$, in which $DC^j(x)$ denotes the Jacobian of the j -th column of $C(x)$ and $(CC^+)^j(x)$ is the j -th column of $(CC^+)(x)$ with $C(x)^+$ defined as the Moore-Penrose pseudoinverse¹ of $C(x)$.

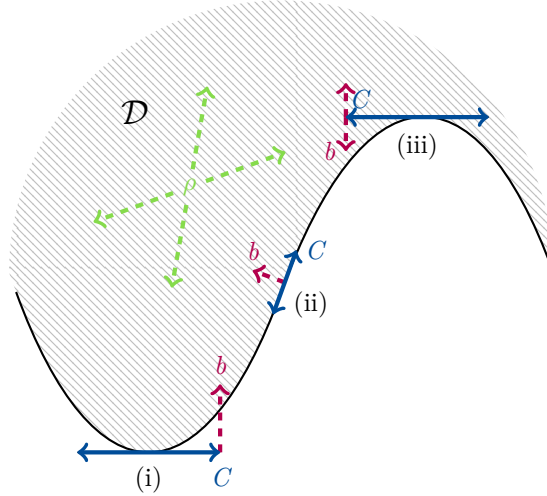


Figure 1: Interplay between the geometry/curvature of \mathcal{D} and the coefficients (b, C, ρ) .

Before moving to the proof, we start by giving the geometric interpretation of conditions (2.1a)–(2.1d), also shown in Figure 1. Condition (2.1c) states that at the boundary of the domain, the column of the covariance matrix should be tangential to the boundary, while (2.1a) requires from \mathcal{D} to capture all the jumps of the process. Moreover, at the boundary, the jumps can have infinite

¹The Moore-Penrose pseudoinverse of a $m \times n$ matrix A is the unique $n \times m$ matrix A^+ satisfying: $AA^+A = A$, $A^+AA^+ = A^+$, AA^+ and A^+A are Hermitian.

variation only if they are parallel to the boundary, by (2.1b). Finally, it follows from (2.1d) that the compensated drift should be inward pointing. We notice that the compensated drift extends the Stratonovich drift (see [9, 13]) when the volatility matrix can fail to be differentiable. In fact, if the volatility matrix is smooth, [1, Proposition 2.4] yields

$$\langle u, \sum_{j=1}^d D\sigma^j(x)\sigma^j(x) \rangle = \langle u, \sum_{j=1}^d DC^j(x)(CC^+)^j(x) \rangle, \quad \text{for all } x \in \mathcal{D} \text{ and } u \in \text{Ker } \sigma(x)^\top.$$

Conversely, the example of the square root process $C(x) = x$ and $\sigma(x) = \sqrt{x}$ on $\mathcal{D} := \mathbb{R}_+$ shows that σ fails to be differentiable at 0 while C satisfies (H₂).

We recall the following crucial lemma for the proof of Theorem 2.2. This is an immediate consequence of the implicit function theorem giving the regularity of the distinct eigenvalues of C and their corresponding eigenvectors under (H₂). We refer to [1, Lemma 3.1] for the proof.

Lemma 2.3. *Assume that $C \in \mathcal{C}_{loc}^{1,1}(\mathbb{R}^d, \mathbb{S}^d)$. Let $x \in \mathcal{D}$ be such that the spectral decomposition of $C(x)$ is given by*

$$C(x) = Q(x) \text{diag}[\lambda_1(x), \dots, \lambda_r(x), 0, \dots, 0] Q(x)^\top,$$

with $\lambda_1(x) > \lambda_2(x) > \dots > \lambda_r(x) > 0$ and $Q(x)Q(x)^\top = I_d$, $r \leq d$.

Then there exist an open (bounded) neighborhood $N(x)$ of x and two measurable \mathbb{M}^d -valued functions on \mathbb{R}^d , $y \mapsto Q(y) := [q_1(y) \cdots q_d(y)]$ and $y \mapsto \Lambda(y) := \text{diag}[\lambda_1(y), \dots, \lambda_d(y)]$ such that

- (i) $C(y) = Q(y)\Lambda(y)Q(y)^\top$ and $Q(y)Q(y)^\top = I_d$, for all $y \in \mathbb{R}^d$,
- (ii) $\lambda_1(y) > \lambda_2(y) > \dots > \lambda_r(y) > \max\{\lambda_i(x), r+1 \leq i \leq d\} \vee 0$, for all $y \in N(x)$,
- (iii) $\bar{\sigma} : y \mapsto \bar{Q}(y)\bar{\Lambda}(y)^{\frac{1}{2}}$ is $C^{1,1}(N(x), \mathbb{M}^d)$, in which $\bar{Q} := [q_1 \cdots q_r \ 0 \cdots 0]$ and $\bar{\Lambda} = \text{diag}[\lambda_1, \dots, \lambda_r, 0, \dots, 0]$.

Moreover, we have:

$$\langle u, \sum_{j=1}^d D\bar{\sigma}^j(x)\bar{\sigma}^j(x) \rangle = \langle u, \sum_{j=1}^d DC^j(x)(CC^+)^j(x) \rangle, \quad \text{for all } u \in \text{Ker}(C(x)). \quad (2.2)$$

We will also need the following lemma giving the continuity of the infinitesimal generator of (1.1) acting on smooth functions ϕ

$$\mathcal{L}\phi := D\phi b + \frac{1}{2} \text{Tr}(D^2\phi\sigma\sigma^\top) + \int (\phi(\cdot + \rho(\cdot, z)) - \phi - D\phi\rho(\cdot, z)) F(dz), \quad (2.3)$$

where $D\phi^\top$ (resp. $D^2\phi$) is the gradient (resp. Hessian) of ϕ . A similar formulation in the semimartingale set-up can be found in [21, Lemma A.1]. In the sequel, we denote by $\mathcal{C}(\mathcal{D})$ the space of continuous functions on \mathcal{D} . The superscript p denotes functions with p -continuous derivatives for all $p \leq \infty$, and the subscript c (resp. 0) stands for functions with compact support (resp. vanishing at infinity).

Lemma 2.4. *Under (H_C) and (H₀), $\mathcal{L}(\mathcal{C}_c^2(\mathcal{D})) \subset \mathcal{C}_0(\mathcal{D})$.*

Proof. Let $\phi \in \mathcal{C}_c^2(\mathcal{D})$. We extend it to $\mathcal{C}_c^2(\mathbb{R}^d)$. Let $M > 0$ such that $\phi(x) = 0$ if $\|x\| > M$. Let $\|x\| > M + 1$. Then

$$\mathcal{L}\phi(x) = \int \phi(x + \rho(x, z))F(dz) = \int_{\{\|x + \rho(x, z)\| \leq M\}} \phi(x + \rho(x, z))F(dz).$$

On $\{\|x + \rho(x, z)\| \leq M\}$, $1 + M < \|x\| \leq M + \|\rho(x, z)\|$. Hence, (H_0) yields

$$\begin{aligned} |\mathcal{L}\phi(x)| &\leq \|\phi\|_\infty \int_{\{\|x + \rho(x, z)\| \leq M\}} \frac{\|\rho(x, z)\|^q \ln \|\rho(x, z)\|}{(\|x\| - M)^q \ln(\|x\| - M)} F(dz) \\ &\leq \|\phi\|_\infty L \frac{(1 + \|x\|^q)}{(\|x\| - M)^q \ln(\|x\| - M)}, \end{aligned}$$

where $\|\cdot\|_\infty$ is the uniform norm, which shows that $\mathcal{L}\phi(x) \rightarrow 0$ when $\|x\| \rightarrow \infty$. Moreover, denoting by $\Phi := \int (\phi(\cdot + \rho(\cdot, z)) - \phi - D\phi\rho(\cdot, z))F(dz)$, we have for all $x, y \in \mathcal{D}$

$$\begin{aligned} \Phi(y) &= \int \int_0^1 \int_0^t \rho(y, z)^\top D^2\phi(y + s\rho(y, z))\rho(y, z) ds dt F(dz) \\ &= \int \int_0^1 \int_0^t \rho(y, z)^\top D^2\phi(x + s\rho(y, z))\rho(y, z) ds dt F(dz) \\ &\quad + \int \int_0^1 \int_0^t \rho(y, z)^\top (D^2\phi(y + s\rho(y, z)) - D^2\phi(x + s\rho(y, z))) \rho(y, z) ds dt F(dz) \\ &=: I_1(x, y) + I_2(x, y). \end{aligned}$$

$I_2(x, y) \rightarrow 0$ when $y \rightarrow x$, since $D^2\phi$ is uniformly continuous (recall that ϕ has compact support). In addition, it follows from (H_c) that $I_1(x, y) \rightarrow \Phi(x)$ when $y \rightarrow x$, which ends the proof. \square

Lemma 2.4 highlights the role of the growth condition (H_0) . In fact, (H_1) would only yield that $\mathcal{L}\phi$ is bounded. This is not enough to apply [12, Theorem 4.5.4] to prove that our condition is sufficient, see below.

We can now move to the proof of Theorem 2.2. We follow the proof of [1] and we use the same conditioning/projection argument combined with the techniques of [22] for the jump component.

Proof of Theorem (2.2). Part a. We first prove that our conditions are necessary. Let X denote a weak solution starting at $X_0 = x$ such that $X_t \in \mathcal{D}$ for all $t \geq 0$. If $x \notin \partial\mathcal{D}$, then $\mathcal{N}_{\mathcal{D}}(x) = \{0\}$ and there is nothing to prove. We therefore assume from now on that $x \in \partial\mathcal{D}$. Let $0 < \eta < 1$. Throughout the proof, we fix ψ_η a bounded continuous function on \mathbb{R}^d such that $\psi_\eta = 0$ on $B_\eta(x)$ and $\psi_\eta \rightarrow \mathbf{1}_{\{\mathbb{R}^d \setminus \{0\}\}}$ for $\eta \downarrow 0$, where $B_\eta(x)$ is the open ball with center x and radius η .

Step 1. We start by proving (2.1a). Let $\epsilon > 0$ and $\phi_\epsilon : \mathbb{R}^d \mapsto [0, 1]$ be \mathcal{C}^2 such that $\phi_\epsilon = 0$ on $\mathcal{D} \cup B_\epsilon(x)$ and $\phi_\epsilon = 1$ on $(\mathcal{D} \cup B_{2\epsilon}(x))^c$. \mathcal{D} is stochastically invariant, hence $\phi_\epsilon(X_t) = 0$, for all $t \geq 0$. Since ϕ_ϵ is twice differentiable and bounded, Itô's formula [15, Theorem I.4.57] yields

$$\int_0^t \mathcal{L}\phi_\epsilon(X_s) ds + \int_0^t D\phi_\epsilon(X_s)\sigma(X_s)dW_s + (\phi_\epsilon(X_{s-} + \rho(X_{s-}, \cdot)) - \phi_\epsilon(X_{s-})) * (\mu - \nu) = 0,$$

where $*$ denotes the standard notation for stochastic integration with respect to a random measure (see [15]) and $\nu(dt, dz) := dtF(dz)$. By continuity of $\mathcal{L}\phi$ (see Lemma 2.4), taking the expectation, dividing by t and letting $t \rightarrow 0$ yield

$$\mathcal{L}\phi_\epsilon(x) = 0. \tag{2.4}$$

A change of probability measure with respect to the Doléans-Dade exponential $Z := \mathcal{E}(\psi_\eta * (\mu - \nu))$, which is uniformly integrable (see [18, Theorem IV.3] and the proof of [22, Proposition 2.13]), yields

$$\int_0^t \tilde{\mathcal{L}}\phi_\epsilon(X_s) ds + \int_0^t D\phi_\epsilon(X_s)\sigma(X_s)dW_s + (\phi_\epsilon(X_{s-} + \rho(X_{s-}, \cdot)) - \phi_\epsilon(X_{s-})) * (\mu - \tilde{\nu}) = 0, \tag{2.5}$$

where

$$\begin{aligned}\tilde{b} &:= b + \int \psi_\eta(z) \rho(\cdot, z) F(dz), \quad \tilde{\nu}(dt, dz) := dt \tilde{F}(dz), \quad \tilde{F}(dz) := (1 + \psi_\eta(z)) F(dz), \\ \tilde{\mathcal{L}}\phi &:= D\phi \tilde{b} + \frac{1}{2} \text{Tr}(D^2\phi C) + \int (\phi(\cdot + \rho(\cdot, z)) - \phi - D\phi \rho(\cdot, z)) \tilde{F}(dz).\end{aligned}$$

By combining the above with (2.3), taking the expectation in (2.5), dividing by t and sending $t \rightarrow 0$, and invoking once again Lemma 2.4 below, we get

$$\mathcal{L}\phi_\epsilon(x) + \int \phi_\epsilon(x + \rho(x, z)) \psi_\eta(z) F(dz) = 0.$$

It then follows from (2.4) that $\int \phi_\epsilon(x + \rho(x, z)) \psi_\eta(z) F(dz) = 0$ for all $\eta \in (0, 1)$. Sending $\eta \downarrow 0$ leads to $\int \phi_\epsilon(x + \rho(x, z)) F(dz) = 0$, by monotone convergence (recall that $\phi_\epsilon \geq 0$). Hence

$$\int \mathbf{1}_{\{x + \rho(x, z) \in (\mathcal{D} \cup B_{2\epsilon}(x))^c\}} F(dz) = 0.$$

For $\epsilon \downarrow 0$, (2.1a) follows from monotone convergence again.

Step 2. By the proof of [1, Proposition 3.5], it suffices to consider the case where the positive eigenvalues of the covariance matrix C at the fixed point $x \in \mathcal{D}$ are all distinct as in Lemma 2.3. We can also restrict the study to $\sigma = C^{\frac{1}{2}}$ (see [1, Remark 2.1]). We therefore use the notations of Lemma 2.3. We proceed as in Step 2 of the proof of [1, Lemma 3.2] for the continuous part combined with the proof of [22, Proposition 2.13] for the jump part. Fix $u \in \mathcal{N}_{\mathcal{D}}(x)$ and let ϕ be a smooth function (with compact support in $N(x)$) such that $\max_{\mathcal{D}} \phi = \phi(x)$ and $D\phi(x) = u^\top$.² Since \mathcal{D} is stochastically invariant, $\phi(X_t) \leq \phi(x)$, for all $t \geq 0$. Let $w_\eta := (\eta - 1)\psi_\eta$. By reapplying Step 1, with the test function ϕ (resp. w_η) instead of ϕ_ϵ (resp. ψ_η), we obtain

$$\begin{aligned}0 &\geq \int_0^t \tilde{\mathcal{L}}\phi(X_s) ds + \int_0^t D\phi(X_s) \sigma(X_s) dW_s + \tilde{N}_t \\ &= \int_0^t \tilde{\mathcal{L}}\phi(X_s) ds + \int_0^t (D\phi Q \Lambda^{\frac{1}{2}} Q^\top)(X_s) dW_s + \tilde{N}_t,\end{aligned}$$

where $\tilde{N}_s := (\phi(X_{s-} + \rho(X_{s-}, \cdot)) - \phi(X_{s-})) \rho(X_{s-}, \cdot) * (\mu - \tilde{\nu})$ is the pure-jump true martingale part under the new measure (recall that ϕ has compact support). Let us define the Brownian motion $B = \int_0^\cdot Q(X_s)^\top dW_s$, recall that Q is orthogonal, together with $\bar{B} = (B^1, \dots, B^r, 0, \dots, 0)^\top$ and $\bar{B}^\perp = (0, \dots, 0, B^{r+1}, \dots, B^d)$. Since $Q \Lambda^{\frac{1}{2}} = \bar{Q} \bar{\Lambda}^{\frac{1}{2}}$, the above inequality can be written in the form

$$0 \geq \int_0^t \tilde{\mathcal{L}}\phi(X_s) ds + \int_0^t D\phi(X_s) \bar{\sigma}(X_s) d\bar{B}_s + \int_0^t (D\phi Q \Lambda^{\frac{1}{2}})(X_s) d\bar{B}_s^\perp + \tilde{N}_t.$$

Let $(\mathcal{F}_s^{\bar{B}})_{s \geq 0}$ be the completed filtration generated by \bar{B} . Since \bar{B}, \bar{B}^\perp are independent and \bar{B} has independent increments, conditioning by $\mathcal{F}_t^{\bar{B}}$ yields by Lemma A.3 in the appendix

$$0 \geq \int_0^t \mathbb{E}_{\mathcal{F}_s^{\bar{B}}}[\tilde{\mathcal{L}}\phi(X_s)] ds + \int_0^t \mathbb{E}_{\mathcal{F}_s^{\bar{B}}}[D\phi(X_s) \bar{\sigma}(X_s)] d\bar{B}_s.$$

We now apply Lemma A.1 of the Appendix to $(D\phi \bar{\sigma})(X)$ and reapply the same conditioning argument to find a bounded adapted process $\tilde{\eta}$ such that

$$0 \geq \int_0^t \theta_s ds + \int_0^t \left(\alpha + \int_0^s \beta_r dr + \int_0^s \gamma_r d\bar{B}_r \right)^\top d\bar{B}_s, \quad (2.6)$$

²Such a function always exists (up to considering an element of the proximal normal cone), see the discussion preceding [1, Lemma 3.2] and Step 1 of the proof of the same Lemma.

where

$$\begin{aligned}\theta &:= \mathbb{E}_{\mathcal{F}^{\bar{B}}} [\tilde{\mathcal{L}}\phi(X.)] \quad , \quad \alpha^\top := (D\phi\bar{\sigma})(x) = D\phi(x)Q(x)\Lambda(x)^{\frac{1}{2}} \\ \beta &:= \mathbb{E}_{\mathcal{F}^{\bar{B}}} [\tilde{\eta}.] \quad , \quad \gamma := \mathbb{E}_{\mathcal{F}^{\bar{B}}} [D(D\phi\bar{\sigma})\bar{\sigma}(X.)].\end{aligned}$$

Step 3. We now check that we can apply Lemma A.2 below. First note that all the above processes are bounded. This follows from Lemmas 2.3 and 2.4, (H₁) and the fact that ϕ has compact support. In addition, given $T > 0$, the independence of the increments of \bar{B} implies that $\theta_s = \mathbb{E}_{\mathcal{F}_T^{\bar{B}}} [\tilde{\mathcal{L}}\phi(X_s)]$ for all $s \leq T$. From Lemma 2.4 and since X has almost surely no jumps at 0, it follows that θ is a.s. continuous at 0. Moreover, since $D\phi\bar{\sigma}$ is $\mathcal{C}^{1,1}$, $D(D\phi\bar{\sigma})\bar{\sigma}$ is Lipschitz which, combined with (A.4), implies (A.1).

Step 4. In view of Step 3, we can apply Lemma A.2 to (2.6) to deduce that $\alpha = 0$ and $\theta_0 - \frac{1}{2} \text{Tr}(\gamma_0) \leq 0$. The first equation implies that $\alpha^\top \Lambda(x)^{\frac{1}{2}} Q^\top(x) = u^\top C(x) = 0$, or equivalently (2.1c) since $C(x)$ is symmetric. The second identity combined with $D\phi(x) = u^\top$ shows that

$$\begin{aligned}0 &\geq \tilde{\mathcal{L}}\phi(x) - \frac{1}{2} \text{Tr} [\bar{\sigma}^\top D^2\phi\bar{\sigma} + (I_d \otimes u^\top) D\bar{\sigma}\bar{\sigma}] (x) \\ &= \mathcal{L}\phi(x) - \frac{1}{2} \text{Tr} [\bar{\sigma}^\top D^2\phi\bar{\sigma} + (I_d \otimes u^\top) D\bar{\sigma}\bar{\sigma}] (x) + (\eta - 1) \int (\phi(x + \rho(x, z)) - \phi(x)) \psi_\eta(z) F(dz),\end{aligned}$$

in which \otimes stands for the Kronecker product (see [1, Definition A.4 and Proposition A.5]) and $D\bar{\sigma}$ is the Jacobian matrix of $\bar{\sigma}$ (see [1, Definition A.7]). Sending $\eta \downarrow 0$, by monotone convergence, we get

$$0 \geq \mathcal{L}\phi(x) - \frac{1}{2} \text{Tr} [\bar{\sigma}^\top D^2\phi\bar{\sigma} + (I_d \otimes u^\top) D\bar{\sigma}\bar{\sigma}] (x) + \int (\phi(x) - \phi(x + \rho(x, z))) F(dz). \quad (2.7)$$

In particular, since $\phi(x) = \max_{\mathcal{D}} \phi$, (2.1a) implies that $\int |\phi(x + \rho(x, z)) - \phi(x)| F(dz) = \int (\phi(x) - \phi(x + \rho(x, z))) F(dz) < \infty$. Moreover, the right hand side is equal to

$$- \int D\phi(x) \rho(x, z) F(dz) - \int \int_0^1 \int_0^t \rho(x, z)^\top D^2\phi(x + s\rho(x, z)) \rho(x, z) ds dt F(dz),$$

yielding (2.1b) (recall (H₁) and that ϕ has compact support). Combining (2.7), (2.2)-(2.3) and

$$\text{Tr} [(I_d \otimes u^\top) D\bar{\sigma}\bar{\sigma}] (x) = \langle u, \sum_{j=1}^d D\bar{\sigma}^j(x) \bar{\sigma}^j(x) \rangle,$$

we finally obtain (2.1d).

Part b. We now prove that our conditions are sufficient. It follows from (2.1c) and the proof of [1, Proposition 4.1] that

$$\text{Tr}(D^2\phi(x)C(x)) \leq -\langle D\phi(x)^\top, \sum_{j=1}^d DC^j(x)(CC^+)^j(x) \rangle,$$

for any smooth function ϕ such that $\max_{\mathcal{D}} \phi = \phi(x) \geq 0$. Moreover, after noticing that $D\phi(x)^\top \in \mathcal{N}_{\mathcal{D}}(x)$ (this is immediate from the Taylor expansion of ϕ around x), (2.1b) yields

$$\begin{aligned}\int (\phi(x + \rho(x, z)) - \phi(x) + D\phi(x) \rho(x, z)) F(dz) &= \int (\phi(x + \rho(x, z)) - \phi(x)) F(dz) \\ &\quad + \int D\phi(x) \rho(x, z) F(dz).\end{aligned}$$

In addition, it follows from (2.1a) that $\phi(x + \rho(x, z)) \leq \phi(x)$ for F -almost all z . Combining all the above with (2.1d) we finally get

$$\mathcal{L}\phi(x) \leq \langle D\phi(x)^\top, b(x) - \int \rho(x, z)F(dz) - \frac{1}{2} \sum_{j=1}^d DC^j(x)(CC^+)^j(x) \rangle \leq 0.$$

Therefore, \mathcal{L} satisfies the positive maximum principle. In addition, since $\mathcal{L} : \mathcal{C}_c^\infty(\mathcal{D}) \mapsto \mathcal{C}_0(\mathcal{D})$ (see Lemma 2.4) and $\mathcal{C}_c^\infty(\mathcal{D})$ is dense in $\mathcal{C}_0(\mathcal{D})$, by [12, Theorem 4.5.4], there exists a càdlàg $(\mathcal{D} \cup \Delta)$ -valued solution to the martingale problem for \mathcal{L} , where Δ denotes the one point compactification of \mathcal{D} . Δ is attained either by jump (killed by a potential) or by explosion. By the discussion preceding [5, Proposition 3.2], the process cannot jump to Δ . Moreover, the growth conditions (H_1) ensure that no explosion happens in finite time (see (A.4)). Hence Δ is never attained. We conclude by using [17, Theorem 2.3]. \square

3 Formulation in the semimartingale framework

In this section, we provide an equivalent formulation of Theorem 2.2 in the semimartingale set-up which is more adapted to the construction of affine and polynomial jump-diffusions. By the work of [11, 7], (1.1) is a very general formulation, equivalent to the formulation of (special) Itô-semimartingales (see also [3, Section 3.2]).

Let X denote an Itô-semimartingale in the sense of [15, Definition III.2.18] on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$, i.e. its semimartingale characteristics $(\tilde{B}, \tilde{C}, \nu^X)$ are of the form

$$\tilde{B}_t = \int_0^t \tilde{b}(X_s)ds, \quad \tilde{C}_t = \int_0^t \tilde{c}(X_s)ds, \quad \nu^X(dt, dz) = dtK(X_t, dz),$$

with respect to a continuous truncation function h . Here, $\tilde{b} : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\tilde{c} : \mathbb{R}^d \mapsto \mathbb{S}_+^d$, K is a measurable transition kernel from \mathbb{R}^d into $\mathbb{R}^d \setminus \{0\}$ and

$$\tilde{b}, \tilde{c} \text{ and } \int f(z)\|z\|^2 K(., dz) \text{ are continuous for any bounded continuous function } f. \quad (\tilde{H}_C)$$

The triplet $(\tilde{b}, \tilde{c}, K)$ is called the differential characteristics of X . In addition we assume that there exist $\tilde{q}, \tilde{L} > 0$ such that

$$\int_{\{\|z\| > 1\}} \|z\|^{\tilde{q}} \ln \|z\| K(x, dz) \leq \tilde{L}(1 + \|x\|^{\tilde{q}}), \quad (\tilde{H}_0)$$

$$\|\tilde{b}(x)\|^2 + \|\tilde{c}(x)\| + \int \|z\|^2 K(x, dz) \leq \tilde{L}(1 + \|x\|^2), \quad (\tilde{H}_1)$$

for all $x \in \mathbb{R}^d$. It follows that X is a special semimartingale. Recall that ν^X is the compensated measure of the random jump measure μ^X of X . By [15, Theorem II.2.38], the special semimartingale X admits the following canonical decomposition

$$X_t = X_0 + B + X^c + z * (\mu^X - \nu^X), \quad (3.1)$$

where X^c is a continuous local martingale with quadratic variation $\langle X^c \rangle = \int_0^\cdot \tilde{c}(X_s)ds$ and $B := \int_0^\cdot b(X_s)ds$, where $b := \tilde{b} + \int (z - h(z))K(., dz)$. Finally, we assume that

$$\text{the restriction of } \tilde{c} \text{ to } \mathcal{D} \text{ can be extended to a } C_{loc}^{1,1}(\mathbb{R}^d, \mathbb{S}^d) \text{ function,} \quad (\tilde{H}_2)$$

and we denote by C this extended function.

We are now ready to state an equivalent formulation of Theorem 2.2 adapted to (3.1).

Theorem 3.1. Let $\mathcal{D} \subset \mathbb{R}^d$ be closed. Under the continuity assumptions (\tilde{H}_C) and (\tilde{H}_0) -(\tilde{H}_2), the set \mathcal{D} is stochastically invariant with respect to the semimartingale (3.1) if and only if

$$\begin{cases} \text{supp } K(x, dz) \subset \mathcal{D} - x, \end{cases} \quad (3.2a)$$

$$\begin{cases} \int |\langle u, z \rangle| K(x, dz) < \infty, \end{cases} \quad (3.2b)$$

$$\begin{cases} C(x)u = 0, \end{cases} \quad (3.2c)$$

$$\begin{cases} \langle u, b(x) - \int z K(x, dz) - \frac{1}{2} \sum_{j=1}^d DC^j(x)(CC^+)^j(x) \rangle \leq 0, \end{cases} \quad (3.2d)$$

for all $x \in \mathcal{D}$ and $u \in \mathcal{N}_{\mathcal{D}}(x)$.

Proof. By [7, Theorem 3.33], after eventually enlarging the probability space, \mathcal{D} is stochastically invariant with respect to (3.1) if and only if it is invariant with respect to (1.1) with

$$\tilde{b} = b - \int (z - h(z))K(., dz), \quad C = \sigma\sigma^\top \text{ on } \mathcal{D} \quad \text{and} \quad K(x, dz) = F(\rho(x, .) \in dz). \quad (3.3)$$

Hence, it suffices to prove that the assumptions of Theorem 2.2 are equivalent to those of Theorem 3.1 and that (2.1a)-(2.1d) are equivalent to (3.2a)-(3.2d). This is immediate from the last identity in (3.3). In fact, $\int g(z)K(x, dz) = \int g(\rho(x, z))F(dz)$, for any measurable function g . In addition, the following identity

$$K(x, \mathcal{D}^c - x) = \int F(dz)\mathbb{1}_{\mathcal{D}^c - x}(\rho(x + z)) = \int F(dz)\mathbb{1}_{\mathcal{D}^c}(x + \rho(x + z)),$$

shows that (2.1a) is equivalent to (3.2a). \square

We end this section with a remark when X is not necessarily special.

Remark 3.2. We can easily get a similar result when X is not necessarily special by truncating the jumps. This would be equivalent to studying the invariance with respect to the following stochastic differential equation

$$\begin{aligned} dX_t = & b(X_t)dt + \sigma(X_t)dW_t + \int h(\rho(X_{t-}, z))(\mu(dt, dz) - F(dz)dt) \\ & + \int (\rho(X_{t-}, z) - h(\rho(X_{t-}, z)))\mu(dt, dz), \end{aligned}$$

instead of (1.1). In this case all the jump terms appearing in the invariance conditions of Theorems 2.2 and 3.1 should also be truncated, e.g. (3.2b) and (3.2d) would read

$$\int |\langle u, h(z) \rangle| K(x, dz) < \infty \quad \text{and} \quad \langle u, \tilde{b}(x) - \int h(z)K(x, dz) - \frac{1}{2} \sum_{j=1}^d DC^j(x)(CC^+)^j(x) \rangle \leq 0.$$

A Technical lemmas

The generalized Itô's lemma derived in [1, Lemma 3.3] can easily be extended to account for jumps in the following way.

Lemma A.1. Assume that σ is continuous and that there exists a solution X to (1.1). Let $f \in \mathcal{C}_c^{1,1}(\mathbb{R}^d, \mathbb{R})$. Then, there exists an adapted bounded process η such that

$$f(X_t) = f(x) + \int_0^t \tilde{\eta}_s ds + \int_0^t (Df\sigma)(X_s)dW_s + (f(X_{s-} + \rho(X_{s-}, z)) - f(X_{s-})) * (\mu - dtF(dz)),$$

for all $t \geq 0$, with $\tilde{\eta} = (Dfb)(X_s) + \eta_s + \int (f(X_s + \rho(X_s, z)) - f(X_s) - Df(X_s)\rho(X_s, z))F(dz)$.

The following adapts [1, Lemma 3.4] to our setting.

Lemma A.2. *Let $(W_t)_{t \geq 0}$ denote a standard d -dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Let $\alpha \in \mathbb{R}^d$ and $(\beta_t)_{t \geq 0}$, $(\gamma_t)_{t \geq 0}$ and $(\theta_t)_{t \geq 0}$ be predictable processes taking values respectively in \mathbb{R}^d , \mathbb{M}^d and \mathbb{R} and satisfying*

- (1) β is bounded,
- (2) $\int_0^t \|\gamma_s\|^2 ds < \infty$, for all $t \geq 0$,
- (3) there exists $\eta > 0$ such that

$$\int_0^t \int_0^s \mathbb{E} [\|\gamma_r - \gamma_0\|^2] dr ds = O(t^{2+\eta}), \quad (\text{A.1})$$

- (4) θ is a.s. continuous at 0.

Suppose that for all $t \geq 0$

$$\int_0^t \theta_s ds + \int_0^t \left(\alpha + \int_0^s \beta_r dr + \int_0^s \gamma_r dW_r \right)^\top dW_s \leq 0. \quad (\text{A.2})$$

Then, $\alpha = 0$, $-\gamma_0 \in \mathbb{S}_+^d$, $\theta_0 - \frac{1}{2} \text{Tr}(\gamma_0) \leq 0$.

Proof. Since $(W_t^i)^2 = 2 \int_0^t W_s^i dW_s^i + t$, (A.2) reduces to

$$(\theta_0 - \frac{1}{2} \text{Tr}(\gamma_0))t + \sum_{i=1}^d \alpha^i W_t^i + \sum_{i=1}^d \frac{\gamma_0^{ii}}{2} (W_t^i)^2 + \sum_{1 \leq i \neq j \leq d} \gamma_0^{ij} \int_0^t W_s^i dW_s^j + R_t \leq 0,$$

where

$$\begin{aligned} R_t &= \int_0^t (\theta_s - \theta_0) ds + \int_0^t \left(\int_0^s \beta_r dr \right)^\top dW_s + \int_0^t \left(\int_0^s (\gamma_r - \gamma_0) dW_r \right)^\top dW_s \\ &=: R_t^1 + R_t^2 + R_t^3. \end{aligned}$$

In view of [4, Lemma 2.1], it suffices to show that $R_t/t \rightarrow 0$ in probability. To see this, first note that $R_t^1 = o(t)$ a.s. since θ is continuous at 0. Moreover, [6, Proposition 3.9] implies that $R_t^2 = o(t)$ a.s., as β is bounded. Finally, it follows from (A.1) that $\frac{R_t^3}{t} \rightarrow 0$ in L^2 , and hence in probability. We conclude by applying [4, Lemma 2.1]. \square

We also used the following elementary lemma which extends [23, Lemma 5.4] to account for jumps (see also [19, Corollaries 2 and 3 of Theorem 5.13]). This is immediate by invoking the independence of the Brownian motion and the Poisson random measure (see [14, Theorem II.6.3]). Notice that the result is still valid for any martingale with independent increments (not necessarily a Brownian motion).

Lemma A.3. *Let B, B^\perp denote two independent Brownian motions and μ a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with compensator $dt \otimes F$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Let $(\gamma_s)_{s \geq 0}$ be an adapted square integrable process and $\rho : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$ be a predictable process such that $\int_0^t \int \|\rho(s, z)\|^2 F(dz) ds < \infty$, for all $t \geq 0$. Define the sub-filtration $\mathcal{F}_t^B = \sigma\{B_s, s \leq t\} \subset \mathcal{F}_t$ and denote by $\tilde{\mu} = \mu - dtF(dz)$. Then \mathbb{P} -a.s., for all $t \geq 0$,*

$$\mathbb{E}_{\mathcal{F}_t^B} \left[\int_0^t \gamma_s dB_s \right] = \int_0^t \mathbb{E}_{\mathcal{F}_s^B} [\gamma_s] dB_s, \quad \mathbb{E}_{\mathcal{F}_t^B} \left[\int_0^t \gamma_s dB_s^\perp \right] = \mathbb{E}_{\mathcal{F}_t^B} [\rho * \tilde{\mu}] = 0.$$

Moreover, it holds similarly for any integrable adapted process θ that

$$\mathbb{E}_{\mathcal{F}_t^B} \left[\int_0^t \theta_s ds \right] = \int_0^t \mathbb{E}_{\mathcal{F}_s^B} [\theta_s] ds.$$

For completeness, we recall well-known moment estimates for (1.1) under (H_1) .

Proposition A.4. *Let X denote a weak solution of (1.1) starting at x . Under the growth conditions (H_1) , there exists $M_{x,L}^1 > 0$ such that the following moment estimates hold:*

$$\mathbb{E} \left[\sup_{s \leq t} \|X_s\|^2 \right] \leq 4 \left(\|x\|^2 + Lt(t+8) \right) e^{4Lt(t+8)}, \quad \text{for all } t \geq 0, \quad (\text{A.3})$$

$$\mathbb{E} \left[\|X_t - X_s\|^2 \right] \leq M_{x,L}^1 |t - s|, \quad \text{for all } s, t \leq 1. \quad (\text{A.4})$$

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